BERA
BRITISH EDUCATIONAL RESEARCH ASSOCIATION

BRITISH CURRICULUM FORUM
BRITISH EDUCATIONAL RESEARCH ASSOCIATION

LEARNING FROM VARIATION

A BCF CURRICULUM INVESTIGATION GRANT REPORT

Ruth Trundley
Helen J. Williams

About BERA

The British Educational Research Association (BERA) is the home of educational research in the United Kingdom. We are a membership association committed to advancing knowledge of education by sustaining a strong and high quality educational research community.

Together with our members, BERA is working to advance research quality, build research capacity and foster research engagement.

Since its inception in 1974, BERA has expanded into an internationally renowned association with both UK and non-UK based members. It strives to be inclusive of the diversity of educational research and scholarship, and welcomes members from a wide range of disciplinary backgrounds, theoretical orientations, methodological approaches, sectoral interests and institutional affiliations. It also encourages the development of productive relationships with other associations within and beyond the UK.

Aspiring to be the home of all educational researchers in the UK, BERA provides opportunities for everyone active in this field to contribute through its portfolio of distinguished publications, its world-class conference and other events, and its active peer community, organised around 33 special interest groups. We also recognise excellence in educational research through our range of awards. In addition to our member-focussed activity, we aim to inform the development of policy and practice by promoting the best quality evidence produced by educational research.

About the British Curriculum Forum

The British Curriculum Forum aims to bring together all those with an interest in collaborative curriculum, research and development.

Through events, awards and grants, the BCF supports communication and collaboration in the study and practical implementation of the curriculum in schools, colleges and wider educational settings. Connecting schools, colleges, universities and others, our work promotes the study of theoretical, innovative and practical aspects of the curriculum, drawing on a rich history, spanning more than 40 years, and continuing the tradition of research and development founded by Lawrence Stenhouse.

The British Curriculum Forum is the successor to the British Curriculum Foundation which was incorporated into BERA in 2014. The BCF has been in existence for over 40 years (and was previously known as the Association for the Study of Curriculum, or Curriculum Association).

The BCF aims to:

- promote the study of theoretical, innovative and practical aspects of the curriculum
- provide an authoritative medium through which the opinions of teachers and others may be expressed on matters of the curriculum
- provide means of communication among all those concerned with the study of the curriculum and/or its practical implementation
- enable BERA to connect with schools
- enable practitioners to engage with research.

About the BERA BCF Curriculum Investigation Grant

BERA and the BCF's biennial Curriculum Investigation Grant is intended to support and recognise the importance of research led by schools and colleges that focusses on curriculum inquiry and investigation. It is awarded to researchers based within schools and colleges, and is intended to enable those researchers to:

- identify an issue impacting on the development of an aspect of the curriculum in their school/college
- design, implement and evaluate a response to the issue identified
- disseminate the processes and outcomes of the inquiry/investigation within the school/college
- develop a strategy to sustain curriculum investigation/inquiry within the school/college
- contribute to research and scholarship in the study of the curriculum.

This report by Ruth Trundley and Helen Williams is one of three projects supported by the Curriculum Investigation Grant for 2018–2019. The other two projects, reports from which have been published simultaneously, were:

- *Local language, school and community: Curricular innovation towards closing the attainment gap*, by Claire Needler and Jamie Fairbairn
- *Exploring task design as an enabler of leading teaching in secondary schools: Practical curriculum development through the use of theory*, by Lorna Shires and Mat Hunter.

For more information about the Curriculum Investigation Grant and these reports, see bera.ac.uk/award/bcf-curriculum-investigation-grant

Contents

Published in January 2020 by the British Educational Research Association

British Educational Research Association (BERA)
9–11 Endsleigh Gardens
London WC1H 0EH

www.bera.ac.uk
enquiries@bera.ac.uk
020 7612 6987

Charity Number: 1150237

Download

This document is available to download from:
https://www.bera.ac.uk/publication/bcf-cig_learning-from-variation

Citation

If referring to or quoting from this document in your own writing, our preferred citation is as follows.

> Trundley, R., & Williams, H. J. (2020). *Learning from variation: A BCF Curriculum Investigation Grant report*. London: British Educational Research Association. Retrieved from https://www.bera.ac.uk/publication/bcf-cig_learning-from-variation

SUMMARY

This research project was one of three winners of the British Curriculum Forum's Curriculum Investigation Grant for 2018–2019. Researchers Dr Ruth Trundley and Dr Helen J. Williams, both with an interest in the development of children's understanding of early number, worked in collaboration with three teachers from two primary schools in south-west England to investigate how variation theory (Marton & Tsui, 2005) might be applied to the use of manipulatives (that is, any objects that can be moved and handled by learners) to support understanding of early number. The project's aims were to increase awareness of current narratives of variation theory and to ascertain how applicable these might be to younger learners of mathematics, as little work has yet been done to explore its application to teaching and learning mathematics in the earlier years.

Variation theory was developed by Marton (with Tsui, 2005), who considered it to be at the heart of learning. It has become dominant in current mathematics discussions, and centres on drawing attention to underlying relationships in mathematics by focussing on the careful design and sequencing of mathematical tasks (Watson, 2016), including the use of multiple representations of a mathematical concept in order to draw out what it is and what it is not.

The project focussed on a selected group of 12 children aged between five and six years old (year 1 in English schooling). The mathematical focus chosen was the move from 'counting-all' to 'counting-on', which research indicates is critical for numerical understanding (Nunes & Bryant, 2009) but which is difficult to reliably establish (Thompson, 2008). Counting-on is the ability to establish the new quantity in a group in which the amount has been increased, without needing to recount the original group. This project explored which manipulatives might be effective in highlighting the essential features of this mathematical idea.

The project had two parallel threads.

- Three teaching sessions with each of four the sub-groups of three children, with one-to-one assessment sessions to begin and end the input, and with concluding individual assessments taking place five months later.
- Ongoing meetings with the teachers involved, taking account of their views – the underlying principle being that we (both researchers and teachers) were collaboratively working *on* rather than working *through* research (Trundley, 2019).

It also had two phases.

- The researchers' initial data collection period and the sharing of these findings.
- The teachers' exploration of a chosen aspect of the findings over the following term, which led to some further findings of interest.

Within this project, grounded theory (Hammersley & Atkinson, 1983) underlay both the design and analysis of data. Both researchers involved have a social-constructivist view of learning (Howe & Mercer, 2007), which informed our decision to use qualitative, semi-structured interviewing with both children and adults throughout in order to allow both space for the voices of individuals involved and sufficient flexibility for the project to take account of our findings as they emerged.

Each researcher worked with two trios of year 1 children, each of whom were identified by their class teacher as being on the cusp of understanding counting-on. Each of the four groups of children used a different manipulative commonly used in classrooms. Assessments were focussed on exploring how the children thought about, understood and made sense of the mathematics rather than just numerical answers.

We found that using the selected manipulatives to aid counting-on was far more complex than we anticipated, and agree with Thompson (2013) that children in year 1 are too young to be expected to be able to count-on reliably. However, the project identified two key sub-skills and understandings that appear to make a significant contribution to children's ability to count-on, and we believe that paying attention to these in Y1 would be valuable. Specifically, these are:

- understanding cardinality and abbreviating the augend (the original quantity)
- keeping track of both location in the number system and of the addend (the number being counted-on) using objects.

Understanding cardinality involves linking the numeral to the set it represents, and appreciating that adding to (or removing from) that set changes the number in the set. *Abbreviating the augend* includes subitising (immediately recognising a small quantity), building on this knowledge to recognise iconic images – such as those contained within structured manipulatives – and making use of this knowledge when counting.

Keeping track of where one is in the sequence of counting words relies on operating fluently within the number system. Keeping track of the addend involves a sophisticated double-count procedure, whereby both the addend and the overall quantity are accurately monitored.

This research makes a substantial contribution to the existing literature on counting-on (see for example Secada, Fuson & Hall, 1983), and has implications for the teaching of counting-on to young children.

References

Hammersley, M. & Atkinson, P. (1983). *Ethnography: Principles in practice.* London: Routledge

Howe, C. & Mercer, N. (2007). *Children's social development, peer interaction and classroom learning: Primary review interim reports.* Cambridge: Cambridge University Press

Marton, F. & Tsui, A. (2005). *Classroom Discourse and the Space for Learning.* Mahwah, NJ: Erlbaum.

Nunes, T. & Bryant, P. (2009). *Key understandings in mathematics learning: Paper 2: Understanding whole numbers.* London: Nuffield Foundation.

Secada, G., Fuson, K. & Hall, W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education*, *14*(1), 47–57.

Thompson, I. (Ed.) (2008). *Teaching and learning early number* (2nd ed.). Maidenhead: Open University Press.

Thompson, I. (2013). Counting: Part 1 – From 'counting all' to 'counting on'. Unpublished paper.

Trundley, R. (2008). The value of two. *Mathematics Teaching*, *211*, 17–21.

Trundley (2019). Professional learning through research: Planning for success and identifying barriers. *Primary Mathematics*, *23*(1), 17–21.

Watson, A. (2016). Variation: Analysing and designing tasks. *Mathematics Teaching*, 252.

1. BACKGROUND TO THE STUDY

During 2019, Dr Ruth Trundley and Dr Helen J. Williams worked in collaboration with two schools – Great Torrington Bluecoat Church of England Primary School in Devon, and St Ives Infant School in Cornwall – to investigate how variation theory (Marton & Tsui, 2005) might be applied to the teaching of early number in year 1 (children aged between five and six years old).

The project had three aims.

- To increase understanding of how variation theory might be applied in relation to younger learners.
- To further understanding of how variation might foster young children's developing mathematics sense.
- To develop pedagogical subject knowledge of the effects of conscious (and unconscious) decisions regarding resource-use.

The project ran during the spring and summer terms of 2019, and involved three teachers in three year 1 classes. The mathematical focus was on the move from 'counting-all' to 'counting-on', which research indicates is critical to numerical understanding (Nunes & Bryant, 2009).

The drivers for this research were twofold. First was increasing awareness nationally of variation theory being key to developing pupils' understanding of mathematical ideas, and a corresponding lack of research into the application of this theory to younger learners which meant that many approaches designed for older learners at key stages 2 and 3 appeared to be surfacing in year 1. Second was awareness that learning is influenced by teacher choices regarding which practical resources to use in order to model mathematical situations. It is well established that manipulatives (that is, any objects that can be moved and handled by learners) are central to helping children develop understandings of mathematical situations (Thompson, 2010; Griffiths, Back, & Gifford, 2016, 2017; Gifford, 2017) because they, among other things, develop visual images and provide a bridge to abstract thinking. Often a wide range of both structured (Numicon©, bead strings) and unstructured (natural objects, beads, counters) manipulatives are used in year 1 mathematics teaching. This project explored which manipulatives are effective at highlighting the essential features of counting-on.

2. RESEARCH DESIGN

This research project was focussed on exploring how the smallest changes in resource usage affect children's learning. Decisions regarding exactly what to use and how to make best use of those resources, teaching session by teaching session, are less well-researched than the use of manipulatives more generally. In order to be most effective, choices regarding, for example, the size and colour of counters to be used should be deliberate and intentional.

Grounded theory (Hammersley & Atkinson, 1983) informed both the design and analysis of data within this project. Research that employs grounded theory sets out to generate theory grounded in the data produced, rather than to verify theory.

Our attention to the effect of small variations in resource usage coincides with a flurry of interest in variation theory: variation has, for instance, been identified by the National Centre for Excellence in Teaching Mathematics (NCETM) as one of their five big ideas of 'teaching for mastery'. Our early research question reflected the theory's newfound prominence.

> *How can variation theory be applied to the use of manipulatives to support understanding of early number?*

Early number is a complex area of mathematics, and we chose to focus on one particular element of it: counting-on. In a review of research literature about how children learn mathematics (Nunes & Bryant, 2009), 'counting-on' was identified as a 'developmental shift'.

> *'…[B]etween the ages of five and seven years, there is a definite developmental shift from counting-all to counting-on: as children grow older they begin to adopt the more economic strategy of counting-on from the previously counted subset. This new strategy is a definite sign of children's eventual recognition of the additive composition of the new set… [T]he developmental change that we have just described does suggest an improvement in children's understanding of additive relations between numbers during their first two years at school.'*
>
> Nunes & Bryant, 2009, pp. 19–20

The decision to focus on counting-on led us to a further decision to focus on year 1 pupils, and from both our review of previous research in this area and our subsequent observations during the teaching sessions we concluded that we should revise our research question.

> *What are the key sub-skills of counting-on, and how can variation with manipulatives be used to support development of these sub-skills?*

This research project was led by two researchers, both with an interest in the development of understanding in relation to early number, who worked in collaboration with year 1 teachers in two schools, the details of which are set out in table 2.1.

Table 2.1
Details of the two schools that participated in this research project

School name	Great Torrington Bluecoat Church of England Primary School	St Ives Infant School
School details	An early years specialist school with over 500 pupils aged 2–11	An infant school with just under 200 pupils aged 3–7
School location	North Devon	South-west Cornwall
Participating year 1 teachers	Chris Dayment Misa Magee	Leone Pulley
Researcher	Dr Ruth Trundley	Dr Helen Williams

Each researcher worked with two trios of year 1 children, who were selected by the class teachers according to the following criteria.

- Can accurately count out up to 14 individual items.
- Are likely to work out the right answer – though not by counting-on – when answering questions such as the following.:
  - *'There are four Christmas presents under the tree. I put another two more there; how many are there under the tree now?'*
  - *'There are three Christmas presents under the tree. I put another five more there; how many are there under the tree now?'*
- Are usually reliable attendees.

Different manipulatives were used with each of the four groups of children, as shown in figures 2.1–2.4: one unstructured resource (counters) and three structured resources.

We chose the three structured resources because they support children in recognising the augend without needing to count. (Throughout this report, we use *augend* to refer to the original quantity to which another amount is added, and *addend* to refer to the number that is added to another.) Counters were chosen as the 'control' in the sense that they are not structured but can be organised and arranged in different ways. All four groups used counters at some point during each session.

Figure 2.1
Counters, used by group 1

Figure 2.2
Numicon, used by group 2

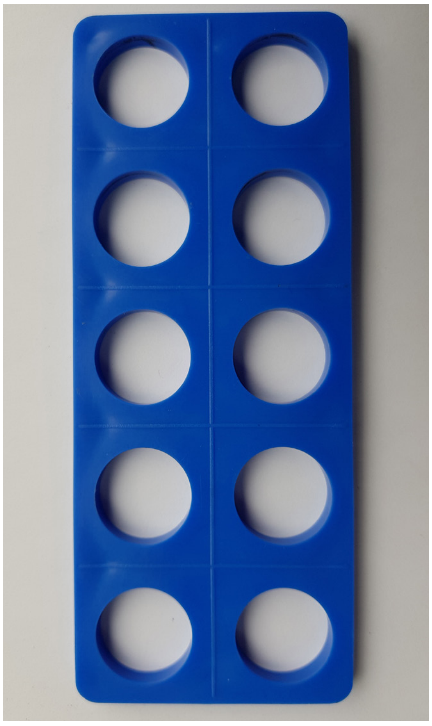


Figure 2.3
Bead strings, used by group 3

Figure 2.4
Ten-frames, used by group 4

The picture book *365 Penguins* by Fromental and Jolivet (2006) was chosen as the context for the mathematics in the sessions because the story has a counting-on structure to it: one more penguin arrives each day.

A deliberate decision was made to involve a small world character, Milo, in these sessions, to allow the children to consider what someone new and external to them was doing and to comment openly on this, free of assumptions and preconceptions that might have been held in relation to classmates. Research into the use of puppets has shown that 'children are keen to talk and explore alternative suggestions' (Keogh & Naylor, 2009), particularly when the puppet needs some help: Milo sometimes made mistakes and was sometimes correct. The use of a small world character, rather than a puppet, meant it was easy for both the researcher and the children to manipulate the character, and he could sit and observe without being intrusive. Although the characters used in the two schools were not identical in appearance, both were named Milo and we planned for them to behave similarly.

Three teaching sessions took place with each group of three children; three one-to-one assessment sessions with each child were also held. The pre-teaching and post-teaching assessments used identical questions, and five months after teaching, the final assessment involved adapted questions (see appendix 1). Table 2.2 outlines the ideas worked on during each teaching session.

Table 2.2
Broad outlines of ideas worked on during each teaching session

Teaching session	Ideas
One	Linking into the story context: representing numbers of penguins with counters. Introducing Milo. Recognising 'five' arranged in different ways. Matching numbers of counters to numbers of penguins. Matching counter arrangements: recognising 'similar' and 'different'. Familiarisation with the structure of a particular manipulative. Using manipulative to represent 'one more than five'.
Two	Counting aloud from and to different, linked starting points. Using manipulative to model 'one more than' different numbers between one and nine. Generalising this to adding one to larger numbers.
Three	Counting aloud from and to different, linked starting points. Adding more than 'one more than': adding 'three more' to different amounts. 'You can add anything onto anything by counting-on.'

Each teaching session was sufficiently flexible for the researchers to respond to the children's thinking, and each concluded with some free drawing and writing about 'what we have learned', in order to gather a wider range of reactions to the teaching.

3. LITERATURE REVIEW

The following areas of research into mathematics learning and teaching were relevant to this study: variation theory, manipulatives and counting. While the existing literature informed our initial research design, because we employed grounded theory our review of relevant literature continued during and after our data collection.

3.1 Variation

Variation is about drawing attention to underlying relationships in mathematics, and thus involves considering both the design and sequencing of tasks (Mason, 2011; Lo, 2012; Watson, 2016). Variation theory incorporates the use of multiple representations of a mathematical concept to draw out what it is and what it is not. It is argued that by paying attention to keeping some things the same while some things change, learners become more able to reason and make connections between what they know (NCETM 2017).

> *'This theory has been developed by Ference Marton (2005) and his co-researchers… He believes that learning only happens if there is some variation to discern and he sees learning as the discernment of variation. Because mathematical concepts are largely concerned with variables and structures, the theory applies fairly easily to learning about mathematical concepts and techniques… we need to consider invariance as well as variation and it turns out to be important to work out how much or how little variation is necessary for learners to notice what we hope they will notice.*
>
> Watson & Mason, 2006, p. 3

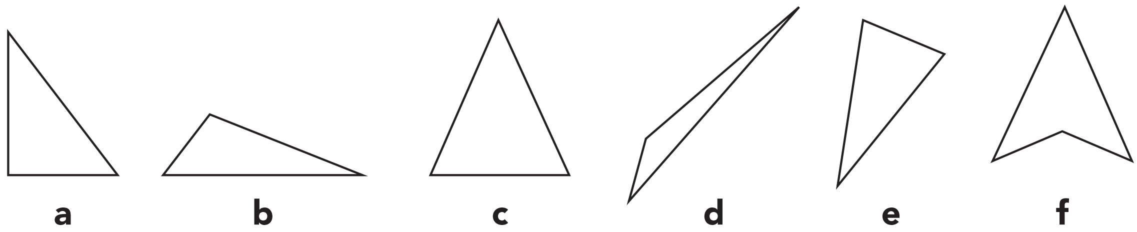
Gu, Huang and Marton (2004) identify two main aspects of variation: the procedural and the conceptual. They identify conceptual variation as helping students understand concepts from multiple perspectives, and procedural variation as helping students to connect what they know already with the object of learning.

Examples of procedural variation focus on varying numbers, as in the following example by the NCETM (2016).

58 – 24 = ____	36 – 25 = ____	53 – 22 = ____	49 – 24 = ____
57 – 25 = ____	46 – 24 = ____	64 – 23 = ____	48 – 25 = ____
56 – 26 = ____	56 – 23 = ____	75 – 24 = ____	47 – 26 = ____

Examples of conceptual variation often focus on contexts and pictures, as in figure 3.1.

Figure 3.1
Examples and non-examples of triangles



Source: NCETM, 2016

Variation theory corresponds with many other theories of learning: its consonance with constructivism, for example, is exemplified in the work of Fosnot and Dolk (2002), which is built around ideas of variation.

Although variation has been identified and disseminated by NCETM as part of their training in 'teaching for mastery', little if any research has yet been published that explores its application to teaching and learning mathematics in the earlier years, and the use of manipulatives. Such a lack may be associated with the fact that statutory schooling in England begins up to two years earlier than is common internationally and, as a result, most of the examples of variation given in the literature – often in the form of written practice questions and examples – are more suited to older learners.

3.2 Manipulatives

Chen, Brownell and Uttal (2019) see the goal of early mathematics education as being to help children make sense of the relationship between concrete materials and the abstract concepts they represent. Detailed decisions regarding exactly which manipulatives to use, and how and when to use them, are not well researched. Teachers can be observed making many complex decisions on a daily basis while teaching number sense, and the smallest changes in resource usage can affect children's learning. Resources must be chosen carefully in order to promote understanding (Chen et al., 2019). However, 'it's true that most teacher decision-making is split second and grounded in perception, feelings, interpretations, and reaction' (Fosnot & Dolk, 2002, p. 146).

3.3 Counting, cardinality and counting-on

Counting is complex and a considerable cognitive achievement for children (MacLellan, 2009). Research into young children's understanding of the ordinal counting sequence stems from the work of Piaget. Piaget (1952) argued that children produce the right words in the correct order long before they understand what these numbers represent. Gelman and Gallistel (1978) described learning to count as including five interrelated principles: the 'one-to-one principle', the 'stable-order principle', the 'cardinal principle', the 'abstraction principle' and the 'order-irrelevance principle'. In order to become fluent counters, children need repeated counting practice over a number of years, in a variety of contexts in which counting is meaningful (Griffiths, Back, & Gifford, 2016). One key link for children to make is that between the number and the quantity it represents, and thus between the order of number words in the counting sequence and the magnitude of the quantity represented (Nunes & Bryant, 2009).

Cardinality has developed as an idea over the past 40 years. It is one of the five counting principles identified by Gelman and Gallistel (1978): 'the last "tag" is also the number name for the whole set'. Sarnecka, Cerrutti and Carey (2005) expanded on this definition by including the 'successor function'.

> *'We suggest that the missing piece may be understanding of the successor function (the function describing how numbers are formed: N, N + 1, [N + 1] + 1..., etc.)... We conclude that researchers should think of children's cardinal-principle knowledge as a last word rule plus understanding of the successor function.'*
>
> Sarnecka et al., 2005, p.1

Trundley (2008) further expanded the meaning of cardinality.

> *'Cardinality... [is] built on an ability to subitise and attach number names to small numbers. It includes recognition of the fact that these small numbers can be partitioned in a variety of ways, including previous number + 1, leading to an understanding of the relationship between successive numbers ("successor function") as well as being able to say the number name for the whole set.'*
>
> Trundley, 2008, p. 20.

Nunes and Bryant suggest that counting-on, as opposed to counting-all, is a significant development. They describe it as 'a sign that the children have linked their knowledge of part–whole relations with the counting sequence: they have understood the additive composition number' (Nunes & Bryant, 2009, p. 4). Fuson's analysis of counting-on (1982) identified four meanings for the single word used for the augend and different structures for keeping track of the addend. Secada, Fuson and Hall (1983) built on this work through research into the use of counting-on in addition, and identified the significance of the sizes of the augend and addend. They also suggested useful teaching strategies to adopt in order to aid the transition from counting-all to counting-on when adding 'm + n': 'when n is more than three the child must have some way of keeping track of how many words beyond m are being produced' (Secada et al., 1983, p. 47)

This research informed our decisions. For example, we decided to keep the addend to three or fewer so that keeping track, which was not the focus of our study, was made easier. As our data emerged, the work of Fuson (1982) and Secada et al. (1983) became increasingly relevant, and informed our data analysis significantly.

4. METHODOLOGY AND CONTEXT

4.1 Beliefs and values of researchers

Carr (1985) states that all educational research is underpinned by the researcher's philosophical beliefs and educational values, and that researchers have a responsibility to identify and justify the values that are embedded in their work. Both researchers involved in this project have a social-constructivist view of learning. Social constructivism questions theories that regard children's intellectual achievements as solely the product of individual discovery, and instead regards knowledge as created and shared among members of communities, with mathematical thinking at its heart, and considers learners to be agents in their own learning (Lerman, 2000; Howe & Mercer 2007).

Decisions made in the teaching sessions and throughout the research reflected these beliefs: children were invited to make decisions, expected to explain their thinking and provided with opportunities to explore and reflect upon their own understandings through open mark-making. Assessments were focussed on exploring how the children thought about, understood and made sense of the mathematics rather than just numerical answers.

4.2 Case study and grounded theory

This project involved a case study of 12 children in year 1, all but one of whom had not yet reached their sixth birthday when the teaching sessions took place, during January 2019.

Table 4.1
Birth month and year of participating children

Birth month & year	Number of children
October 2012	1
February 2013	1
March 2013	1
April 2013	3
June 2013	1
July 2013	2
August 2013	3

Although the participating children were selected according to the criteria set out above in the introduction, they presented with quite different levels of understanding and previous experience in relation to counting and adding.

Grounded theory (Hammersley & Atkinson, 1983) informed both the design and analysis of data within this project. Research that employs grounded theory sets out to generate theory grounded in the data produced, rather than verify theory. Literature is reviewed *after* data collection; it is in fact additional data. Data collection and data analysis exist as two parts of a whole: analysis of the first set of data influences the collection of the second set of data, and so on.

The data collection for this research was structured as follows.

Table 4.2
Dates of and participants in each data collection activity

Date	Activity	Participants
Spring term, week 1	Assessment 1	Individual pupils
Spring term, week 1	Meeting with the children and sharing the text	Trios of pupils
Spring terms, weeks 1 & 2	Three half-hour teaching sessions	Trios of pupils
Spring term, week 2	Assessment 2	Individual pupils
Spring term, week 7	Sharing of data analysis with teachers; teachers identify areas to explore	All teachers and researchers
Summer term, week 8	Assessment 3	Individual pupils
Summer term, week 8	Observation data collection from teachers	All teachers and researchers

In this research project, movement from one data collection activity to another was preceded by analysis of the existing data, with the two researchers working on the analysis in collaboration. For example, the activity with the trios of pupils in the Devon school was held a day or two after the same activity had taken place in the Cornwall school. This meant that not only did the data analysis shape the next activity, but the analysis of data from each school had an impact on the next data-collection activity in the other school.

The starting point for the research was observation of the year 1 children. All the data activities were recorded using video, allowing the researchers to make observations both in real time and of the recorded data. These observations allowed the researchers to explore in depth the children's understandings of the mathematics involved in the tasks set. This led to sorting of data, followed by defining and labelling – which in the case of this research gave rise to the identification of a number of sub-skills related to counting-on. Relevant literature was identified to support the data analysis as the project progressed; as the data was analysed and observations sorted and labelled, existing research literature linked to each label was explored. The data was then re-examined in order to exemplify the definitions and labels given to each of the sub-skills.

4.3 Ethics

The purpose of any study is a combination of both individuals' professional gain and the advancement of knowledge and understanding more generally on behalf of the institution and the wider research community. The headteachers of the schools involved in the study recognised that school staff stood to gain potentially valuable knowledge once our findings were shared. Both schools were willing participants in this study.

Prior to the start of the project, all adults involved were informed about the project and signed consent forms, in line with BERA's *Ethical Guidelines* (2018). Written information was given to the carers of the 12 children involved, and they too signed consent forms. It was made clear on this initial consent form that all sessions would be audio- and video-recorded to aid analysis and write-up, but that no child would be identified in any reports emerging from the research. Carers were contacted at the close of the project and asked to give their consent for sections of the subsequent recordings to be used outside of the school and the project by the named researchers, appropriately and sensitively, for the purposes of professional education and academic study only. Consent was obtained to acknowledge the schools and the teaching staff involved.

All data was processed in accordance with the BERA *Ethical Guidelines* (2018) in terms of our responsibilities to participants, obtaining informed consent, openness, disclosure and privacy. At the end of the project, we offered to organise a meeting to share project findings with interested parties; this offer was taken up in one school.

It was important to us that the voices of the school staff were heard in this project. It was essential that all understood that they were involved in the research and not being told what to do; that they were working *on* research tasks rather than working *through* the research (Trundley, 2019). To that end, all meetings that took place were planned in a way that allowed us to take account of the views of the teachers involved. Our decision to use qualitative, semi-structured interviewing with both children and adults supported our ethical objectives in that it allowed space in which the voices of individual interviewees could be heard. We planned our questions in order to stimulate discussion and send the interviews off in directions thought important by interviewees. Moreover, teachers made decisions about what they were going to work on with their classes during the time between the end of our input (January 2019) and the final meeting to close the project (June 2019). This self-directed work stemmed from the discussion points that had emerged from our research – namely, abbreviating the augend and keeping track – and which are explored in some detail in the following chapter. The teachers' findings are included in section 5.3.

5. FINDINGS

Through analysis of the teaching sessions, we identified sub-skills and understandings that make a significant contribution to being able to count-on. The two key sub-skills and understandings observed, which we will explore here, are:

- understanding cardinality and abbreviating the augend
- keeping track.

In the transcripts that follow, ‘R’ refers to ‘researcher’, and ‘C’ to ‘child’.

5.1 Abbreviating the augend

5.1.1 Understanding cardinality and abbreviation of the augend

It has always been difficult for young children to distinguish clearly between

- the last object counted as five, and
- the whole group of objects as five (Gattegno, 1964).

Understanding cardinality is complex. Within our study we observed that understanding of cardinality is linked to counting-on through the notion of abbreviating the augend to a single number (Fuson 1982). In order to count-on, children must be able to use one number as a label for the augend – that is, to use one number to represent the whole set. This means recognising, understanding and trusting the cardinality of the number for the set that forms the augend, and entails understanding that the separate items in the set are represented by and abbreviated to one number. There is no need to count the separate items in order to know how many there are in the set.

The ways in which both augend and addend were represented was important in supporting the year 1 children participating in our study to make sense of the mathematics. The three structured mathematics resources that we used in the project (Numicon, bead strings and ten-frames) were chosen specifically to support children to abbreviate the augend.

During our study, which focussed on using objects to represent penguins, we identified four different ways of noticing the cardinality of the augend without needing to count, leading to abbreviation:

- perceptual subitising
- recognising iconic representations
- recognising composition
- reading symbols.

Perceptual subitising

Perceptual subitising involves recognising a small number without counting or using other mathematical processes (Clements, 1999).

Figure 5.1
Examples of small groups of objects that children might be expected to subsitise

When children subitise, they can see the value of the group without needing to count, and are able to say how many objects there are within it (abbreviating). Looking at figure 5.1, for example, children may 'see' that there are three beads, three fish, two people and three pawns without having to count. The design of this study excluded exploring subitising with the augend; a deliberate choice was made to, throughout the project, start with numbers which could not be subitised . Use of subitising was evident within recognising composition (see below).

Recognising iconic representations

Recognising iconic representations means recognising the value of the group by virtue of its familiar representation.

Figure 5.2
Three iconic representations of a group of five

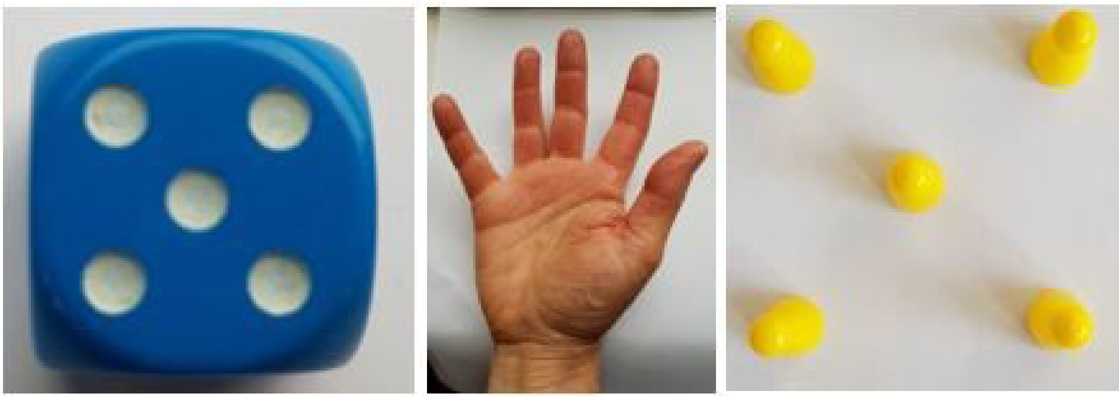


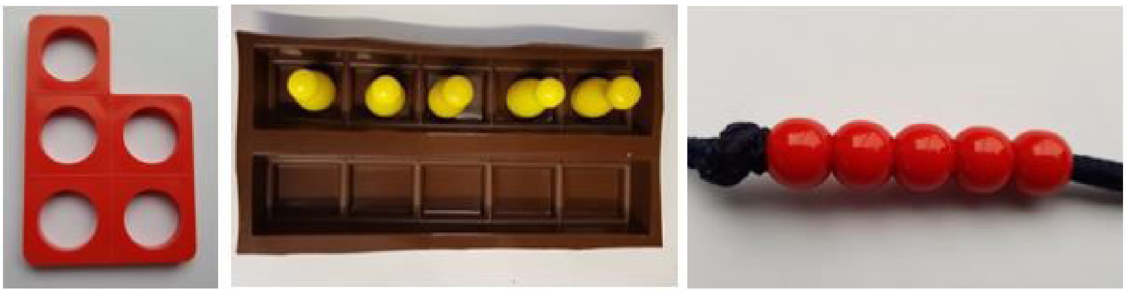
Figure 5.2 shows iconic representations of 'five', including five on a dice and five fingers on one hand. However, iconic representations can also describe familiar arrangements: for example, the pawns in figure 5.2 are arranged like dots on dice.

R How do you know that's five?

C I already know because a five looks like that on a dice.

(Session 1, group 4 [ten-frames])

Figure 5.3
Three iconic representations of 'five' that use structured mathematics resources



Iconic representations also include structured mathematics resources: for example, 'five' can be recognised as a Numicon plate, one complete row of a ten-frame, and one complete block of colour on a 20-bead string, as shown in figure 5.3.

Familiarity with structured resources such as these allows children to notice how many are showing, and to create a set of a given size, without needing to count, as in the following example.

R Can you show me five penguins on the bead string?

C1 That is so easy.

C1 *moves one group of five in one movement.*

R How do you know that's five?

C2 Because you just move one lot and you know that's five.

R Why was that easy?

C1 Because you just needed to take one away… one kind *[referring to one block of colour]*.

R Can you tell Milo why it is easy to find five on the bead string?

C3 Because we know it's five when you have one of the colours.

(Session 1, group 3 [bead strings])

Recognising composition

To recognise composition is to recognise a number by recognising parts of the number. There were many examples in which children saw a number partitioned (usually into two parts), recognised each part (through either subitising or recognising an iconic representation) and knew the number made by the parts together. The children often expressed this using the language of addition.

Having just made 'five' on the bead string the children were asked to make 'seven'.

C That's easy.

R Why is that easy?

C Because I just need to add the two… because five plus two equals seven.

(Session 1, group 3 [bead strings])

Figure 5.4
An example of how the use of a bead string can aid the recognition of composition

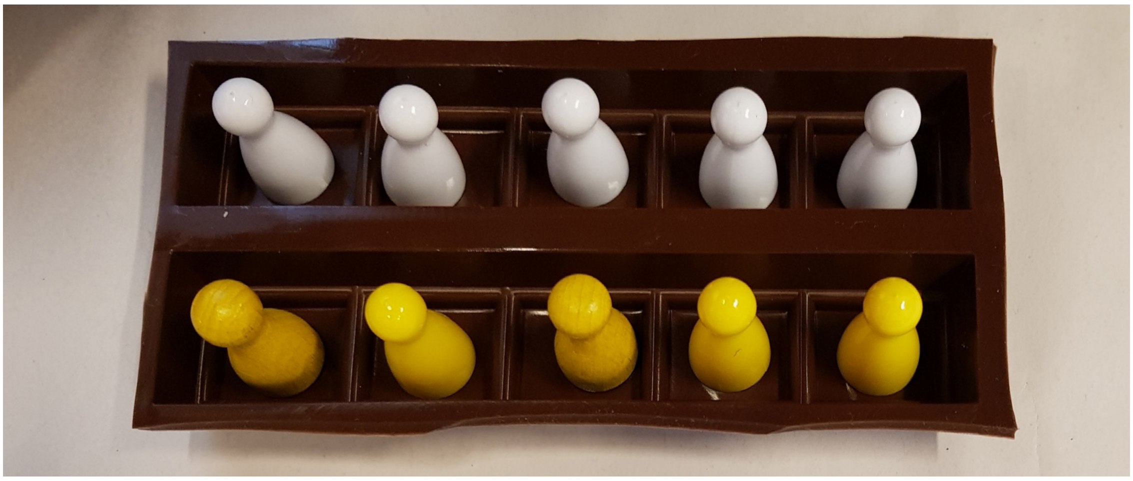
*One child (**C1**) puts a row of white 'penguins' and a row of yellow 'penguins' in the ten-frame.*

R How does [**C1**] know that is ten?

C2 You got five and she got five equals ten.

(Session 1, group 4 [ten-frames])

Figure 5.5
A row of white 'penguins' and row of yellow 'penguins' in a ten-frame



The structured resources supported the children to see how many there were without having to count, and the children were able to use this to abbreviate the augend when focussing on one more. Understanding 'one more' is referred to as the 'successor function' as part of the principle of cardinality (Sarnecka et al., 2005).

We shared the counting-on language that had started to appear in some children's explanations in the context of 'one more'. This was written on the board and used to support the children to think and talk about their thinking, as in the following example.

R Five penguins and one more arrives.

C Six!

R How do you know there will be six now?

C Because 'five, six'; after five is six.

R *writes on the board, 'After 5 is 6'.*

(Session 1, group 4 [ten-frames])

However, that children recognise a representation does not necessarily mean that they will choose to abbreviate the augend and count-on when the number of objects change.

One child recognised that a single block of colour on the bead string represented five.

When two were added he chose to count all the beads in twos.

(Session 1, group 3 [bead strings])

In the context of our study, looking at the use of counting-on to add, in cases in which the augend was small enough to be recognised in one of the three ways described above (that is, perceptual subitising, recognising iconic representations or recognising composition), counting-on was not appropriate because the additions either involved applying a known fact or the children could 'see' the total from their representation of both augend and addend.

In order to move the children to generalise, both about abbreviating the augend and connecting the addend by counting-on, we realised that although the addend needed to be small (we chose to restrict it to three or less) the augend needed to be larger so that the addition involved an unknown number fact. We decided that while we still wanted to have objects representing the augend, these needed to be presented in a way that did not require understanding of place value and that could not be easily counted (counting would take too long), forcing a need for an alternative, more efficient method. This led to the fourth way of recognising cardinality leading to abbreviation.

Reading symbols

In the context of this study we introduced a transparent tub of counters with a number label attached. The children used the label as an abbreviation for the augend.

Figure 5.6
An example of the use of symbols (a number label in this case) as an abbreviation of the augend

Tub with label '45'. Milo counts three more penguins and says it is forty-seven.

C *(looks at the tub)* Forty-five. (*Looks at the counters.*) Forty-six, forty-seven, forty-eight.

R What did Milo do wrong?

C We already counted this number. *[Points at the '45' label.]*

(Assessment 2)

The importance of using and reading symbols arose from the data collected. We suggest that it is fundamental to understanding cardinality: it allows children to understand that a set being labelled '45' (for instance) means that, if all the elements of the set are counted, the count would go from one to 45, meaning that there is no need to count the set. As one child said in her second assessment, in reference to a tub labelled 24 (see figure 5.7): 'How about we get all the counters out: there'll be twenty-four'. This understanding does not require an understanding of the place value of the number involved, or even an ability to count accurately. Rather, it is about understanding the cardinal value of the number.

Figure 5.7
Another tub labelled to indicate the number of counters it contains

Fuson (1982) pointed out that when children do not understand the cardinality of the augend, errors occur when counting-on, as they are only attending to the word sequence (see section 5.2 below, on 'keeping track'). Having counters in a transparent tub supported the children in understanding that the number was more than a number in a count: it was representing a set (which they could see) that was being added to. Looking at the tub when thinking about the augend was key to the children successfully adding to larger numbers by counting-on.

Linked to this is the importance of the abbreviation of the augend, and of how that abbreviation is used when counting-on. Sometimes the children paused and then counted-on without saying the abbreviation for the augend.

An '11' card is on show.

R There are eleven penguins and one more comes along.

C Twelve.

R How do you know it will be twelve?

C Because I counted off of eleven.

(Session 2, group 3 [bead strings])

In other instances, children stated the name on the label and then counted-on.

An '11' card is on show.

R: Ding dong – one more arrives. How many are there?

C Eleven, twelve.

(Session 2, group 4 [ten-frames])

The critical observation here is that it is not about either saying or not saying the abbreviation for the augend. Rather, it is about what the child understands about it when they are saying it: are they making it part of the count when counting-on, or are they acknowledging that it is the name of the set that already exists and then counting-on? During the study children were observed shifting from using the name of the augend as just part of the count to using it to establish the cardinality of the augend. This shift was essential to understanding the abbreviation of the augend and how to use this in order to add by counting-on.

5.2 Keeping track

In the course of our analysis of the teaching sessions it became clear that an understanding of the pattern of the number sequence was significant to the ability to count-on, and that our children often found this challenging. While this might seem obvious, in order to count-on from different starting points children needed to be aware of 'where they were' within the number sequence, rather than simply the number that came 'next'. An additional necessary skill, called upon simultaneously, was the ability to keep track of exactly how many more we were counting-on, and thus when to stop counting – we refer to this here as 'keeping track of the addend using entities'.

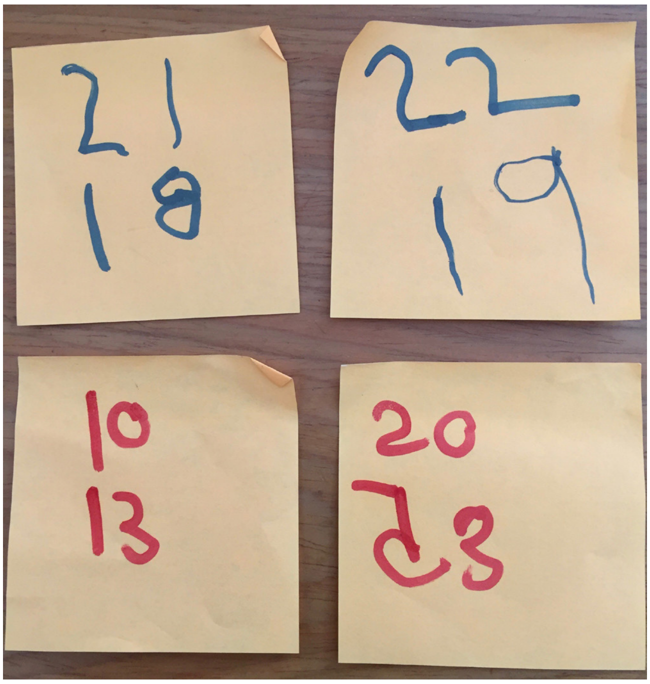
These two related skills are discussed here under the heading 'keeping track'. We will first discuss the ordinal aspect of counting, before considering the issue of how the ordinal and cardinal aspects of counting-on interrelate.

5.2.1 Keeping track of position within the number system

Numbers and quantities are not entirely synonymous (Nunes & Bryant, 2009). While numbers spoken or written in order – ordinal numbers – indicate the position of a quantity in the counting sequence, they can be generated without knowledge of the amount they represent. In order to count at all, a child has to remember some words in the correct order and then be able to access the patterns in our number system in order to generate others. To produce any counting sequence, they need to recognise the aural patterns: for example, *sixty*-one, *sixty*-two, *sixty*-three and sixty-*one*, sixty-*two*, sixty-*three*, as well as the symbolic/visual link between 1, 2, 3 and 61, 62, 63. Moreover, in order to count-on, a child needs to keep track of exactly *where they are* within the counting sequence – that is, once they know the number to start counting from, such as twenty-four, they also need to know that the words to be said next will be twenty-something, as well as something-five, something-six, and so on. Thus, they must be aware of and make use of two patterns simultaneously.

We documented a number of occasions in which children were able to operate with some aspects of the pattern of the number sequence, but not all of them. For example, the children were asked to choose a number from a selection written on the board, write their chosen number on a sticky note, and count-on three (see figure 5.8).

Figure 5.8
Sticky notes produced by children asked to choose a number and then count-on three



In the top-left sticky note shown in figure 5.8, the child appeared to be operating with 18 as eight:

C5 I still don't know what number I choosed [sic]. I still don't know what this number is.

R You chose eighteen.

Pause.

C5 Eleven. I think it's eleven.

R You do? So shall we try it together? *Eight*een…

C5 *mouths quietly alongside as we move three counters.*

R …*nine*teen, *twenty*, twenty-one. Twenty-one!

C5 Twenty-one.

(Session 3, group 2 [Numicon])

On the same occasion a different child counted-on three from 11 as 'twelve, thirteen, forty', confusing the similar sounding 'fourteen' and 'forty'. Another child (in session 3, group 1 [counters]) read the number on the cover of the book *365 Penguins* and counted-on: 'three hundred and sixty five, three hundred and forty five'. This child knew to keep some counting words constant, but changed the wrong numeral. Such mistakes are hardly surprising given both the complexity of our counting sequence and the age of the participating children.

We argue that the ability to generate the counting sequence from differing starting points is to some degree distinct from an understanding of the quantity that each successive number represents, as described by Nunes and Bryant (2009). Gattegno (1970) writes of learners becoming aware of auditory features – 'regularities' of numbers such as *six* and *sixty* – and of noticing patterns which, with practice, generate more numbers in the counting sequence.

> *'I discover that I need, indeed, only a small number of sounds and a small number of principles in order to be as good as my elders in uttering the first ninety-nine numerals.'*
>
> Gattegno, 1970

Counting aloud in unison, starting and stopping in different places and discussing the 'start' and 'stop' numbers was a teaching strategy we introduced in the second and third teaching sessions in order to draw the children's attention to both the pattern of the number sequence and where they were within it. The counting strings that we chose were linked: for example, we started at five and stopped at 12, then started at 25 and stopped at 32, before starting at 105 and stopping at 112. 'Start' and 'stop' numbers where written on a whiteboard.

While planning, it quickly became apparent to us that it was difficult to select linked sequences of consecutive counting numbers that clearly exposed the aural/oral pattern of the counting sequence. Although we needed to cross over 10, we made a deliberate decision to avoid oral counts residing between 11 and 19, due to both lack of regularity (eleven, twelve) and the mismatch between its observable regularity ('1' followed by '5' in '15'), and its orality (fif-teen), although in the event children chose to count from these numbers themselves. A further irregularity was that numbers 'twenty' and 'thirty' – number names early in the counting sequence – are not as transparent as those later in the sequence, such as 'six-ty', 'seven-ty' 'eight-ty' and 'nine-ty'.

All irregularities are difficult to completely avoid. We decided to count aloud including the sequence of 'twenties' and 'thirties', and to vocally emphasise the *ones*-digit rather than the tens-digit, by saying 'twenty-*five*, twenty-*six*, twenty-*seven*' and so on in order to draw children's attention to the familiar changes in the ones digit while the tens digit remained constant within each decade.

Milo was the character who 'told us' which numbers to start counting from and to. When he 'jumped' into the air we were to stop counting. Children, once asked, were quickly able to name the numbers at which we stopped and started, and keen to see how numbers were written as well as to write these numbers themselves.

Milo *'whispers' to* **R**.

R: Forty-seven. Forty-seven he wants us to start at. Shall I write forty-seven up?

Nods and some laughter.

C2 What number did we stop at?

R We *stopped* at sixteen and we *started* at seven, now he wants us to start at *forty*-seven.

C1 Four.

R (*writing)* Four-ty-seven.

C2 Oh that's a big number!

(Session 3, group 2 [Numicon])

During group 2, session 2, the children were shown a '7' numeral card and asked which Numicon plate 'shows one group of seven penguins': all three children immediately selected a 7-plate. They were then asked, 'What if – ding dong – one more penguin comes along, one more? How many penguins would there be?'

C3 It's going to be eight, 'cos after seven is eight

R What if there are *twenty*-seven penguins and one more comes along?

C3 There's still eight

R How do you mean, still eight?

C3 Eight, seven, eighteen

R What do you [C6] think?

C6 Twenty-eight.

R What do you think about what [C6] said?

C3 Twenty-eight, 'cos twenty, eight. When you get your twenty, it was twenty-seven, add on one.

(Session 2, group 2 [Numicon])

Recognising and utilising the patterns underlying the structure of our number system is key as it allows us to count on (and back) from any place without requiring understanding of the place-value (make-up) structure of a number. However, in order to count-on exactly three more from 24, for example, it is not enough for children simply to say the correct count words ('twenty-five, twenty-six, twenty-seven') in the correct order – they also need to know when to *stop* saying these words. They must keep track of how far they are along this particular sequence of counting words.

5.2.2 Keeping track of the addend using entities

This is where the ordinal and cardinal aspects of counting-on interrelate. It is an extension of an earlier counting stage, in which our counting words are matched one-to-one, using the hand or finger, with the object to be counted. In order to keep track of how many we have counted so far, what we say must 'keep up' with our finger movement and recognise that the last one we touch also names the quantity of the entire counted group (cardinality).

Keeping track of the addend while counting-on combines several different skills:

- saying the right words in the right order
- starting the count other than from one or zero
- not 'seeing' (that is, imagining) the original quantity (augend) to be counted
- 'marking' in some way the amount being counted-on (addend)
- understanding that the last number said denotes the final quantity or position.

When counting-on three from seven, I might state 'I have seven', hold up or imagine three fingers, and say, 'Eight, nine, ten. That's ten'. In other words, although I am holding up three fingers I am not saying 'one, two, three', nor am I naming the cardinal value of the group as three. This is sophisticated, as the child is required to match two different strings while not beginning their count at one. Moreover, the count does not match the quantity observed. We would argue that any counting-on employs a 'double counting' procedure (Thompson 2010).

Baroody (1987: 142) categorises using fingers or manipulatives to represent the cardinal value of (at least) the addend as *concrete counting*. Our study focussed on having the number to be counted-on, up to three, represented by objects. Between the four groups, we varied the objects used to denote penguins being added, one at a time, to a known amount. We also documented examples in which children attempted matching the count to their fingers, with varying degrees of success.

Counters (group 1)

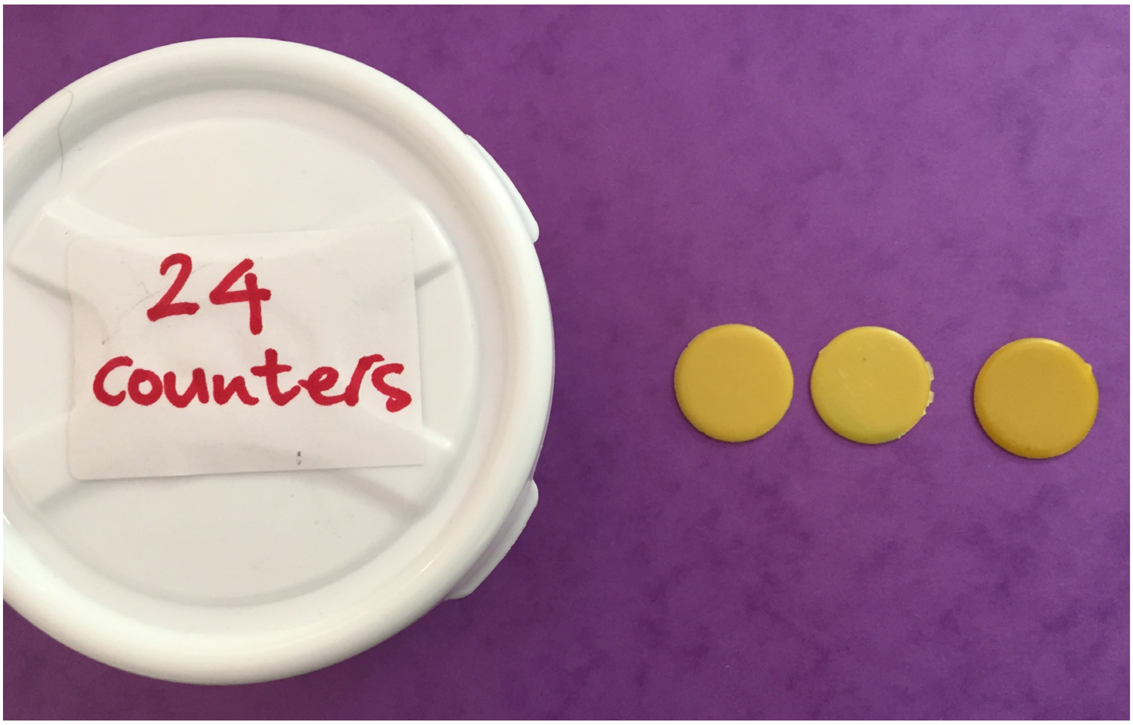
In order to separate and distinguish between the augend and addend, each child had one colour of counter for each. Counters allowed each child choice in how they arranged the augend without having to recount these, and 'to quickly see how many are there'. This drew on their subitising knowledge. During session 1, one child had her plate of seven yellow counters arranged as a dice-six with one in the middle, with three green counters alongside. She said, 'We've got seven, and three more makes ten. We've got seven and we've got *[touching the three yellow counters one at a time]* eight, nine, ten'. With the children having made these arrangements themselves, there was possibly less need to recount.

We also tried two ways of adding on counters with all four groups:

- laying out the addend one at a time while saying the counted-on number words (for example, 'twenty-five, twenty-six…') as they were placed
- counting out the addend ('one, two, three') before then counting-on three from (for example) twenty-four (see figure 5.9).

Both of these approaches were successful with smaller addends when the researcher was demonstrating. Laying counters one at a time requires a child to grasp when to stop, by keeping track of how many they have laid out while saying the count-on number sequence. This is more sophisticated than first placing the whole amount to be added on and then using these to count-on, because at the end of the row of counters you stop saying the number words. However, when counting-on larger addends it is more difficult to keep track of the correct amount if additions are made one at a time. During teaching session two, having found that children were keen to write their own numbers, we decided to write a selection of numbers on the board and invite them to choose a number, write this on a sticky note and to count-on three from their chosen number (this later developed into 'choose your own number to write down', at which point the numbers they counted-on from became much larger). The group using the counters all chose to first count out the second quantity in total each time, before using these as markers to count on from their chosen augend. The choosing and writing of numbers of their own was very successful in terms of their engagement, their ability to count-on a small amount to these larger numbers, and their risk-taking in experimenting with much larger numbers than we would have selected for them.

Figure 5.9
Twenty-four counters in a labelled tub, with three loose counters, representing the augend and addend respectively



Both reading and writing two-digit numbers involves matching how we say and how we write the digits, without necessarily needing to understand the place value of the numbers at this stage. It entails generalising the known sequence of numerals 0 through 9.

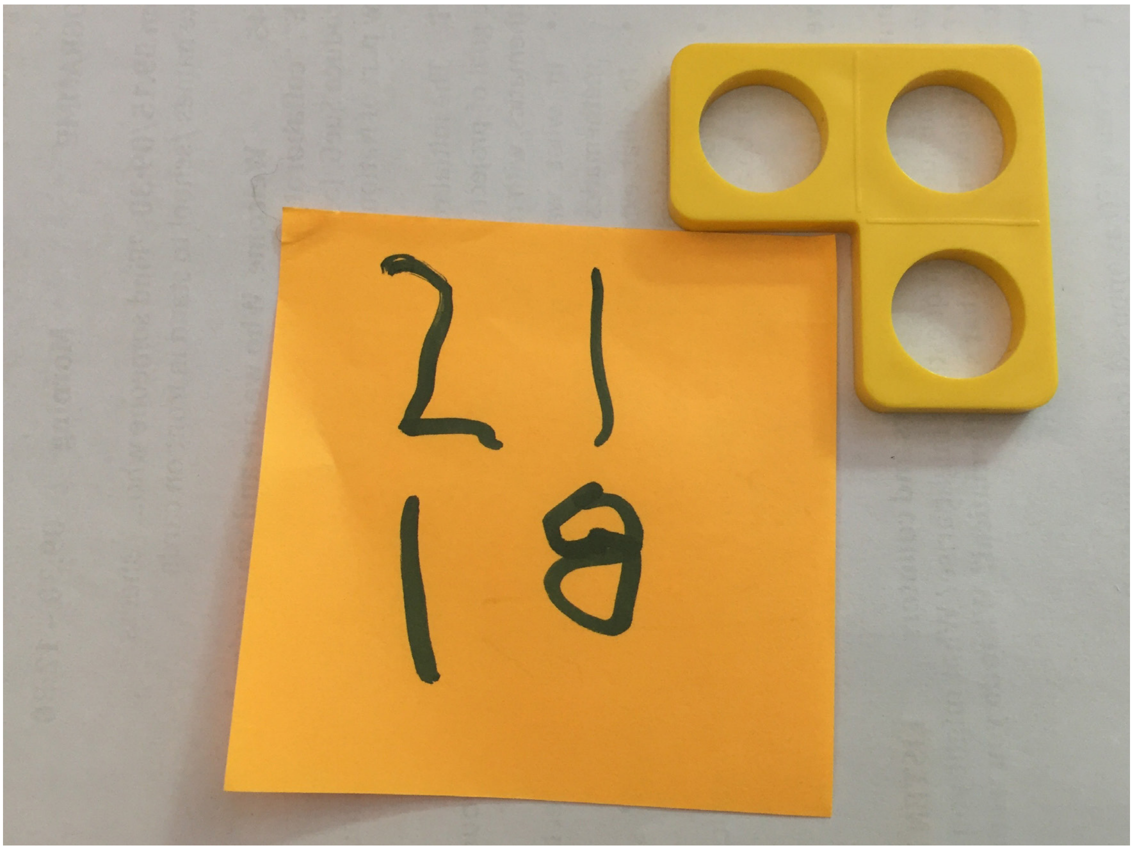
Numicon (group 2)

None of the three structured resources were useful for representing the augend when it was larger than 20, as for two-digit numbers all three rely on an understanding of place value, which was not our focus, whereas counters rely on an understanding of counting.

Of the three structured resources, only Numicon was useful for representing the addend in order to keep track when adding on to larger numbers, because the three-plate maintains its 'threeness' wherever it is placed (next to a tub of counters, for example) whereas both the bead string and the ten-frame involve more than three even when representing three, which adds unnecessary complexity to the image.

During session 3, each child in group 2 took one Numicon three-plate to represent the addend 'count-on three', then chose and recorded a two-digit number from which to count on. This was successful, as the Numicon kept the addend constant while the augend changed.

Figure 5.10
An example of the use of a Numicon three-plate to represent an addend of 3



5.2.3 Use of fingers

The children's level of success using fingers depended on the sizes of the addend and augend, as well as their confidence with and knowledge of the counting sequence. Generally speaking, and unsurprisingly, the larger the numbers involved the more problematic it was keeping track using their fingers.

The task is to add three more to five.

R Can you show us what you did with your fingers?

C6 *(Holds up right hand with all five fingers raised.)* Five.

C6 *holds up left fist and unfolds fingers one at a time while counting.*

C6 Six, seven eight. Eight!

C5 That's exactly what I did!

R We don't need to count these five fingers do we?

(Session 3, group 2 [Numicon])

A little later in this session the children were asked how many penguins there would be if there were 12 already and three more came along. Two children (C6 and C3) both held up their right-hand thumbs.

C6 Fifteen.

R Thirteen.

All *(rhythmically and without reference to their fingers) Fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty*

They all stop.

None of the children managed to keep track using their fingers on this occasion as they were still focussed on reproducing the counting sequence through the teens.

During the second assessment, C6 was asked, 'What is two and seven more?' She held up her right hand and three fingers on her left, touched the 'last' finger and said 'That's… wait, is it seven more?'

R Two and seven more

C6 Oh, Ok.

C6 *holds up two fingers on her left hand, then pauses. She puts up one finger at a time while counting in ones, ending up by holding both hands in the air.*

R How many do you make that?

C6 *(Looks at her hands which are now showing nine fingers)* Two, two, *(quietly counts one to seven nodding at each adjacent finger in turn)* So that will be two plus, *(quietly)* three, four five, six, wait! Have I counted? I can't even count these now!

R Shall we do it the other way around? What is **seven** and **two** more?

C6 Yes.

C6 *counts up seven fingers, adds another two and holds up nine fingers. She does this twice and has to count all before reaching her answer, nine.*

C6 That was really confusing for me then.

While fingers can be a useful marker for keeping track with small amounts to count-on, the efficiency of their use relies on being able to distinguish which fingers denote the augend and which the addend.

5.3 Further findings

The teaching decisions made and the main findings to date were shared with all the participating teachers at the same time, by video link, in week 7 of the spring term. They were provided with a summary sheet (see Appendix 2) and invited to take an aspect of the work completed so far and explore it further in their classes. Each teacher then spent a term exploring their chosen aspect and, at a meeting in the summer term week 8, provided detailed feedback. This led to some further interesting findings, set out in table 5.1.

Figure 5.11
An example of children exploring the structure of numbers through writing prompted by the teacher modelling

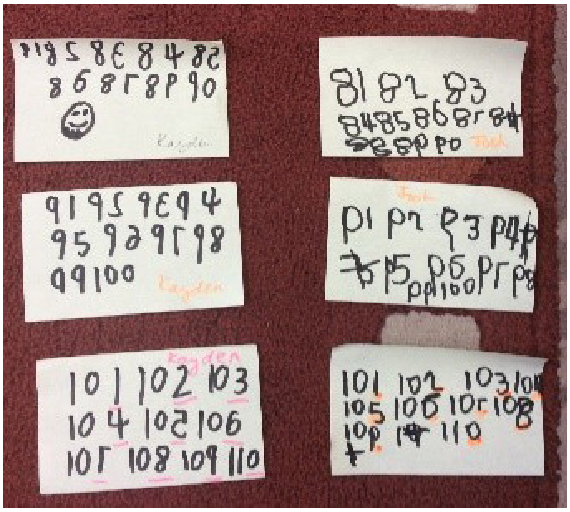


Table 5.1
A summary of the focusses and findings of the three participating teachers during summer term 2019

Teacher	Focus	Findings
Teacher 1	Use of small world character:	Encouraged explanation.
	for playing games	Allowed children to play game with character as a partner.
		Useful for exploring position and direction: a moving character makes it easier to make sense of movements, as its position can be viewed from above.
	for sometimes getting things wrong	Children are less worried about having a go and making mistakes when moving the character.
	for the children to have someone to explain things to in order to help them.	Motivated children.
Teacher 2	Cardinal value	Sometimes the choice of resource for counting can distract the children from the mathematics (colourful dinosaurs, for instance).
	Symbolic representation	Writing the numbers (teacher and children) to match the counts had an impact on children's understanding of the structure of the numbers, and allowed them to make connections to things they know (for example, connecting '3 + 7 = 10' to '43 + 7 = 50') (see figure 5.11).
	Use of small world character	Need to connect context, image, language and symbols.
		Language is linked to cardinal value and trust. 'Do you trust me, that there are seventeen apples, no more, no fewer?'
Teacher 3	Counting aloud from and to different starting points	Increased confidence and accuracy in counting through larger numbers, across decades and from different starting points.
	Cardinal value	Use of labelled containers leading to increased interest in and awareness of written numerals.
	Use of small world character to count, correctly and incorrectly	Increased awareness of the processes of counting, willingness to both correct and explain.

6. DISCUSSION AND CONCLUSIONS

We embarked on this study in order to attain a better understanding of how variation theory (Marton & Tsui, 2005) might be applied to the teaching of early number in year 1 (children aged between five and six). We approached this question with the idea that variation in the type of manipulative used would expose and highlight the essential features of a mathematical idea for the children involved, as well as variation in the choice and sequencing of numbers used. In the event, our attending to variation, and to what it might mean for working with the youngest learners, highlighted some essential features of counting-on and how teaching decisions have an impact on children's understanding of this concept. Attending to the variation in counting-on – keeping something the same and changing something else – allowed us to better understand what is involved in counting-on and, significantly, different ways of noticing the cardinality of the augend. Our findings in relation to the two key sub-skills involved in counting-on arose from our use of variation.

We appreciate that variation is not the doing or the construction of written exercises, nor simply varying the manipulative used. Rather, it is how teachers and learners operate with all representations of the mathematics, considering what is the same and what is different in order to locate underlying structure. For us, the following issues lie at the heart of every theory in teaching.

- What is the intended understanding?
- What are the examples, and what are they examples of?
- What are we going to do and say, and to what purpose?

We found that using the selected manipulatives to aid counting-on was far more complex than we anticipated. For instance, some resources may be unhelpful if, in order to use them to support counting-on, they require an understanding of mathematics that the children have not yet secured: using bead strings, ten-frames or Numicon with larger augends requires children to understand the place value of the number in order to represent the augend. Counters, on the other hand, were more useful than we had anticipated as they required an understanding of counting rather than place value, so when working with larger numbers they could be used to represent and abbreviate a larger augend without the children having to understand its place-value structure. Our prepared transparent tub containing 24 counters, closed and labelled with the correct number of counters inside, appeared to play a central role in supporting children to realise that the symbolic label represented a group that, if counted, would involve saying the numbers from one to twenty-four, and that therefore they did not need to count them.

Figure 6.1
The transparent, labelled tub containing 24 counters, which proved a valuable resource

The counters 'disappeared' into the tub, becoming contained as one quantity; the label stood as an abbreviation for the count of the group, and the children's focus appeared to move to the addend. The group of counters had become unitised and given one label; the children could see the group (that is, they could see that there were lots of items contained within the tub), and that the label represented the number in the group.

However, we cannot be sure whether every child realised that if they were to count the counters in this tub, the number written on the label would be the last number they said.

We developed this idea of 'abbreviating' the augend to the counters not being present, by introducing the practice of writing the augend on a sticky note, and counting-on up to three from there. Children were able to successfully count-on in this way, enthusiastically opting for much larger numbers (in the hundreds) when given the choice, even when they found this problematic.

Although both researchers suspected that counting-on was difficult before starting work on this project, neither of us – despite having many years of experience between us – had fully appreciated its complexity. In particular, we did not appreciate the importance of children being able to abbreviate their counting by naming the augend and then being able to generalise this as the initial quantity in all circumstances and with all numbers, realising that in doing so they are naming the cardinal value of the augend.

Secada et al. (1983) successfully taught seven-year-olds who still used count-all as their main strategy three sub-skills that the authors argued underpinned counting-on. These were:

- continuing the count from an arbitrary point
- switching from the cardinal meaning to the counting meaning of the addend (a cardinal-count transition) (see Thompson, 2008, p. 99)
- beginning the count of the addend with the next counting word.

We recognise these, and tend to agree with Thompson (2013) that children in year 1 are too young to be expected to be able to count-on reliably. However, we believe that paying attention to our two identified sub-skills in year 1 would be valuable. Specifically, these sub-skills are:

- understanding cardinality and abbreviating the augend
- keeping track of
  - position in the number system
  - the addend using entities.

Understanding cardinality and abbreviating the augend, and keeping track of the addend, includes the following.

- Understanding the meaning of a number in relation to a set (cardinal value), that it represents the total number in the set, and that adding to the set changes the number in the set.
- Subitising (recognition) and building on subitising knowledge (composition) towards recognition of iconic images (including structured manipulatives).

Keeping track of position in the number system is about knowledge of the counting sequence, which includes the following.

- Counting aloud from and to different numbers, making connections between number sequences. Children do not need to know the place value of 45 (as formed by four tens and five ones) in order to count-on from 45, for example.
- Identifying 'start' and 'stop' numbers in counting sequences.
- Reading and interpreting two-digit numbers.

REFERENCES

Baroody, A. J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers*. New York: Teachers College Columbia University

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large-scale cases. *Mathematics Education Research Journal, 16*(3), 27–57.

British Educational Research Association [BERA] (2018). *Ethical guidelines for educational research* (4th ed.). London. Retrieved from https://www.bera.ac.uk/researchers-resources/publications/ethicalguidelines-for-educational-research-2018

Carr, W. (1985). Philosophy, values and educational science. *Journal of Curriculum Studies*, *17*(2), 119–132.

Chen, J,. Brownell, J. O., & Uttal, D. (2019). The use of concrete objects in early mathematical learning. In J. S. McCray, J. Chen, & J. Eisenband Sorkin (Eds.), *Growing mathematical minds: Conversations between developmental psychologists and early childhood teachers.* New York: Routledge

Clements, D. H. (1999). Subitising: What is it? Why teach it? *Teaching Children Mathematics, 5*(7), 400–405

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals and percents.* Portsmouth, NH: Heinemann.

Fromental, J. L., & Jolivet, J. (2006). *365 Penguins*. New York: Abrams Books for Young Readers.

Fuson, K. (1982). An analysis of the counting-on solution procedure in addition. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 67–81). Mahway, NJ: Lawrence Erlbaum Associates.

Gattegno, C. (1964). *The Cuisenaire Gattegno method of teaching mathematics: A course for teachers* (Vol. 1). Reading: Educational Explorers Ltd.

Gattegno, C. (1970). Notes on a new epistemology. In L. Brown, D. Hewitt, & D. Tahta (Eds.), *A Gattegno Anthology*. Derby: Association of Teachers of Mathematics.

Gelman, R. & Gallistel, C. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Griffiths, R., Back, J., & Gifford, S. (2016). *Making numbers: Using manipulatives to teach arithmetic.* Oxford: Oxford University Press.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders*. Singapore: World Scientific Publishing.

Hammersley, M. & Atkinson, P., (1983). *Ethnography: Principles in practice.* London: Routledge.

Howe, C., & Mercer, N. (2007). *Children's social development, peer interaction and classroom learning: Primary review interim reports*. Cambridge: Cambridge University Press.

Keogh, B. & Naylor, S. (2009). Puppets count. *Mathematics Teaching*, *213*, 32–34.

Kirkman, J. (2014). *Learning study: The potential of learning study to enhance initial teacher education and continued professional development at the University of Birmingham*. Retrieved from https://www.birmingham.ac.uk/Documents/college-social-sciences/education/research-centres/U21-learning-study-report.pdf

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives in mathematics teaching and learning*. Westport, CT: Ablex Publishers

MacLellan, E. (2009). Counting: What it is and why it matters. In I. Thompson (Ed.), *Teaching and Learning Early Number* (2nd Ed.). Oxford: Oxford University Press.

Marton, F. & Tsui, A. (2005). *Classroom discourse and the space for learning*. Mahwah, NJ: Erlbaum.

Mason, J. (2011). Explicit and implicit pedagogy: Variation theory as a case study. Smith, C. (Ed.) *Proceedings of the British Society for Research into Learning Mathematics*, *31*(3), 107–112.

Lo, M. L. (2012). Variation theory and the improvement of teaching and learning. *Gothenburg Studies in Educational Sciences*, 323.

National Centre for Excellence in Teaching Mathematics [NCETM] (2016). *Big ideas in mastery: Variation*. Sheffield. Retrieved from https://bbomathshub.org.uk/wp-content/uploads/Primary-Variation-Example-Subtraction-and-Triangles.pdf

National Centre for Excellence in Teaching Mathematics [NCETM] (2017). *Supporting research, evidence and argument*. Sheffield. Retrieved from https://www.ncetm.org.uk/resources/50819

Nunes, T., & Bryant, P. (2009). *Key understandings in mathematics learning, paper 2: Understanding whole numbers*. London: Nuffield Foundation.

Piaget, J. (1952). *The origins of intelligence in children* (2nd ed.). New York: International Universities Press.

Sarnecka, B, Cerrutti, A., & Carey, S. (2005). *Unpacking the cardinal principle of counting: A last-word rule + the successor function*. Poster presented at Fourth Biennial Meeting of the Cognitive Development Society, San Diego, CA, 20–21 October 2005. Retrieved from https://studylib.net/doc/8192258/unpacking-the-cardinal-principle-of-counting

Secada, G., Fuson, K. & Hall, W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education*, *14*(1), 47–57.

Thompson, I. (Ed.) (2008). *Teaching and learning early number* (2nd ed). Maidenhead: Open University Press.

Thompson, I. (Ed.) (2010). *Issues in Teaching Numeracy in Primary Schools* (2nd ed.). Maidenhead: Open University Press.

Thompson, I. (2013). Counting: Part 1: From 'counting all' to 'counting on' (unpublished paper).

Trundley, R. (2008). The value of two. *Mathematics Teaching*, *211*, 17–21.

Trundley, R. (2019). Professional learning through research: Planning for success and identifying barriers. *Primary Mathematics 23*(1), 17–21.

Watson, A. (2016). Variation: Analysing and designing tasks. *Mathematics Teaching*, *252*, 13–18.

Watson, A., & Mason, J. (2006). Variation and mathematical structure. *Mathematics Teaching*, *194*, 3–5.

APPENDIX 1
OPENING AND CLOSING ASSESSMENTS

Opening assessment

Resources:

- list of children identified by teacher with dates of birth
- 2 bags of single colours of counter, 2 different colours
- 1 container of 24 counters labelled "24 counters"
- sheet of A1/2 paper
- recording task sheet for each child.

Researcher responses: only use, 'Good job', 'You're trying hard', 'Thank you'.

Warm up: 'I am going to be working with you for the next couple of weeks and I am interested in what you are thinking about numbers.

Today I am going to ask you a few questions. Now, OK, let's start!'

Time taken to answer = 'instant', 'hesitate', 'laboured', 'can't answer'.

	Task / question	Strategy	Time	Answer
1	Can you fetch me 6 counters to put in this dish? (counters of one colour from a dish) How many counters do you have there?	Have to recount them?		
2	(Put the 6 counters in a horizontal line) How many counters do you have there? *(add 3 more counters of a different single colour)* Now how many counters are there altogether?	Have to recount them? Count all to answer?		
3	*(Remove all counters, but leave them available)* If I had 3 cakes on a plate and I add another 5 cakes, how many cakes are on the plate all together?	Do they get 3 quickly? Have to recount them? Count all to answer?		
4	*(Remove all counters, but leave them available. Produce a labelled jar of 24 single-coloured counters)* Here is a jar of 24 counters. I have counted them all and I know there are 24 here. Look there's a label saying 24. Now I am going to add 3 more (produce 3 in a different colour), How many will there be in the jar?	What do they do? Count all? Are the 3 the last 3 they count?		
5	*(Remove all counters and props)* So, what is 5 and 3 more?			
6	What is 2 and 7 more?			

Concluding assessment

Resources:

- list of children identified by teacher with dates of birth
- three tubs, counters in each tub the same colour, with labels 24, 45, 32
- one bowl of counters, single colour, different colour to tubs
- Milo
- sheet of A1/2 paper
- recording task sheet for each child.

Researcher responses: only use, 'Good job', 'You're trying hard', 'Thank you'.

Warm up: 'Hello – how have you been getting on?'

Time taken to answer = 'instant', 'hesitate', 'laboured', 'can't answer'.

	Task / question	Strategy	Time	Answer
1	Here is a tub of 24 counters. Look there's a label saying 24. How about we get all the counters out, how many would there be?			
2	So we've got our tub of 24 counters. Now I am going to add 3 more (place 3 in a different colour by the tub), How many will there be altogether?	What do they do? Count all? Are the 3 the last 3 they count?		
3	*(Switch tubs.)* Milo has a tub of 45 and counts three more (three more are set out) 45, 46, 47 (Milo jumps as he counts). Do you agree with him?			
4	Here is a tub of 32 counters. If you added five more how many would there be altogether?			

APPENDIX 2
OBSERVATIONS FROM TEACHING SESSIONS

Sub-skills of counting on

1. Seeing separate items as one quantity – with any size of group. Working on accepting cardinal values of larger numbers as well as small numbers.
2. Becoming familiar with the patterns in our number system by oral counting up and back from different places within 100.
3. Practice reading and writing two-digit numbers.
4. Keeping track.

Key teaching decisions

Small world character

The introduction of another person into the group, who might not understand, who might get things right or wrong, provides opportunities for the children to explain and to focus on what is involved in counting-on.

Resource choices

The children do not need to understand the cardinal value of 45 in order to count on from 45; resources that are unstructured (counters, for example) can be used to represent a number without having to understand the place-value structure of the number by putting them in a labelled pot. Using a second colour of counter for the second quantity helps to focus on counting-on; placing these counters singly encourages counting-on and is different to placing them in one go or trying to work it out without three objects to count.

Number choices

This includes the choice of starting numbers and finishing numbers when counting, and drawing attention to these. Choosing connected counting strands – such as 7–16, 47–56 or 147–156 – allowed the children to notice and make connections. In addition situations, a larger augend and a small addend encouraged the children to focus on counting-on. Decisions were made regarding when to write numbers to match the maths, when to ask the children to write numbers, and when not to write numbers.

Language choices

Asking 'what if...?' links to variation. What if there were 35 penguins? What if it was cats instead of penguins? What if *four* more came along? There was an expectation that the children would explain their thinking, asking questions such as, 'How do you know?', 'How could we find out?', 'Do you agree with Milo? Why?'. Repeating/echoing the children's own language was also used to support the whole group to think.

Asking the children to make decisions

This is about application of understanding, allowing children to test out their thinking, to move towards generalising and to demonstrate understanding. This included the children using sticky notes for suggesting numbers for Milo to count on from; free writing and drawing at the end of the sessions; and the suggestion of stories to match given numbers.

Creating familiarity

Establishing familiarity with context, resources, small world characters and free writing and drawing is necessary if the children are to be able to attend to the mathematics.

BERA
BRITISH EDUCATIONAL RESEARCH ASSOCIATION